Iteration and Preservation

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Abstract

How are your opinions on a supposition related to your unconditional opinions? One simple answer is *Material Coincidence*: when you are not sure that not q, you are sure that p on the supposition that q just in case you are sure that either not q or p ($\Diamond q \supset (\Box^q p \equiv \Box(q \supset p))$). I give a novel argument against *Material Coincidence*: given weak side-conditions, it entails the implausible claim that being sure implies being sure that you are sure ($\Box p \supset \Box\Box p$).

1 Introduction

My topic will be conditional mental states such as knowing, believing, or intending *conditional on a supposition*. Conditional mental states have been widely employed in philosophy, but their nature is not well understood.[1]

Philosophers often assume that your conditional mental states arise from hypothetically adding the supposition to your stock of knowledge, and minimally changing your unconditional mental states in response. My target will be a principle that many think flows from this idea:

> **Material Coincidence (Mat).** $\Diamond q \supset (\Box^q p \equiv \Box(q \supset p))$
> If you're not sure that not q, then you're sure given q that p just in case you're sure that either not q or p.

I focus on *being sure* because I worry that *belief* is too weak to satisfy **Mat**[2], but my argument is schematic and generalises to other mental states such as knowledge. Since psychological hiccups can make you violate almost any general principle about sureness, I should really talk about what you *should* be sure of, but I will often run with sureness for readability.

[1] Applications include belief revision (Alchourrón et al., 1985; Stalnaker, 2009), defeat (Stalnaker, 2006; Dorst, 2019), imagination and mind-reading (Currie & Ravenscroft, 2002, §1.3, §2.4; Goldman, 2006, ch. 7; Stich & Nichols, 2000), decision theory (Joyce, 1999, ch. 6-7; Ramsey, 1931 [1926]), conditionals (Edgington, 1995; Williamson, 2020), "constrained" attitude ascriptions (Blumberg & Holguín, 2019; Blumberg & Lederman, 2020), judging what others ought to do (Gibbard, 2003, 48ff.), and shared agency (Velleman, 1997).

[2] See Pearson (2024) and Fang (ms) for relevant discussion.

I think **Material Coincidence** is false. My argument against it will rely on a parallelism between conditional mental states and their unconditional counterparts. Consider belief for a moment. Unconditional and conditional belief must have something in common that makes them both beliefs. Whichever way unconditional beliefs connect to desires and intentions, aim at truth, and are constrained by logic, conditional beliefs should, in the same way, connect to conditional desires and conditional intentions, aim at conditional truth, and be constrained by logic. Conditional mental states obey "parallel" constraints to their unconditional counterparts.

The instance of parallelism I need concerns the relationship between first-order and second-order sureness. For unconditional sureness, I reject the **4** axiom but accept $\mathbf{5^c}$, the converse of the **5** axiom:

> **4**. $\Box p \supset \Box\Box p$
> If you are sure that p, you are sure that you are sure that p.

> $\mathbf{5^c}$. $\Box\neg\Box p \supset \neg\Box p$
> If you are sure that you are not sure that p, you are not sure that p.

If conditional and unconditional sureness obey parallel constraints, then if $\mathbf{5^c}$ is true, then so is the parallel principle for conditional sureness:

> $\mathbf{C5^c}$. $\Box^q\neg\Box^q p \supset \neg\Box^q p$
> If you are sure given q that you are not sure given q that p, you are not sure given q that p.

However, we can prove **4** from $\mathbf{C5^c}$ and **Mat** in a normal background logic. I argue that, since **4** is false and $\mathbf{C5^c}$ is true, we must reject **Mat**.

Encountering this argument out of the blue, you should be suspicious. What's so special about **4**-failures that should force you to violate **Mat**? Are there other violations of **Mat**? How do we model conditional sureness without **Mat**? And what about the arguments for **Mat**? In the second half of the paper, I'll try to answer these questions.

2 Motivating Mat

The first order of business is to provide some intuition for what conditional mental states are, and why one might take them to satisfy **Mat**.

Start with some examples. The following utterances would naturally be interpreted as reporting my conditional mental states:[3]

(1) If Snow let the door slam, then I'm sure she left in a hurry.
(2) If Bar hated the restaurant, then I regret recommending it to her.

[3] See Blumberg & Holguín (2019), Blumberg & Lederman (2020), and Holguin (2022). Though see Drucker (2017, forthcoming) for a different take.

(3) If Tez got a hedgehog, then I'm surprised she didn't tell me.

I don't know whether the antecedents of these conditionals are true. Either way, I'm not *unconditionally* sure that Snow left in a hurry, I don't *unconditionally* regret recommending the restaurant, and I'm not *unconditionally* surprised that Tez didn't tell me. The mental states ascribed instead seem to be *conditional* — reporting my state of mind on the supposition that the relevant antecedent is true.

Another way to latch onto conditional mental states is through analogies. Ramsey (1931 [1926], 170) suggested that your conditional opinions are whatever stands to choices between conditional bets as unconditional opinions stand to choices between unconditional bets.

Opinions : Bets :: Conditional Opinions : Conditional Bets

He took a conditional bet to behave like its unconditional counterpart, except that it is void if the supposition fails. The same strategy can of course be applied to other candidate functional roles for belief than choosing bets, such as aiming at truth, or being regulated by the evidence. The challenge then becomes to carve out a "parallel" role for conditional beliefs.

Analogies like this can sometimes be used to argue for **Mat**. The classic such argument starts from a behaviourist conception of how credences are related to bets (and so conditional credences to conditional bets), and shows that on pain of exposing yourself to sure losses, your conditional credences must satisfy the ratio formula ($P(p \mid q) = P(p \wedge q)/P(q)$ when $P(q) > 0$). **Mat** would then follow by a (to my mind, problematic) identification of sureness with credence one. Other arguments of this sort assume connections between rational opinions and accuracy or evidence.[4]

Another way to get a handle on conditional mental states would be to reduce them to unconditional mental states. Two prominent version of this strategy have been pursued.[5] The first identifies conditional mental states with mental states with conditional contents. To be sure of p conditional on q is to be sure that if q then p ($\Box^q p \equiv \Box(q > p)$). **Mat** then follows from *Modus ponens* ($p > q \vdash p \supset q$) and *Embedded Or-to-If* ($\Box(p \vee q) \wedge \neg\Box p \supset \Box(\neg p > q)$), the principle that if you are sure that p or q, without being sure that p in particular, then you are sure that if not p, then q.[6] Instances of *Embedded Or-to-If* sound great: If you're sure that Turin is either in Switzerland or in Italy, and you're not sure it's is in Switzerland, then you must be sure that it's in Italy if not Switzerland![7]

[4] See Greaves & Wallace (2006) and Smith (2018).

[5] Stalnaker (1984, 103) voices both: "To be disposed to accept *B* on learning *A* is to accept *B* conditionally on *A*, or to accept that if *A*, then *B*."

[6] Assume $\Box$ is normal. Then $\Box(q > p) \supset \Box(q \supset p)$ by *MP* and *Normality*. $\Diamond q \supset (\Box(q \supset p) \supset \Box(q > p))$ by *Embedded Or-to-If*. Taking both together, $\Diamond q \supset (\Box(q > p) \equiv \Box(q \supset p))$.

[7] See Stalnaker (1975), Boylan & Schultheis (2022). For critical discussion, see Holguín

Another reduction identifies conditional mental states with plans or dispositions to adopt unconditional mental states upon learning (exactly) the supposition. If you must *plan* or *be disposed* to update by the ratio formula,[8] **Mat** would again follow if sureness was credence one. Other dispositionalists endorse **Mat** as capturing *minimal change*: your new opinions must differ no more from your old ones than is required to consistently include what's learnt while preserving closure under logical consequence.[9] Of course, minimal change will still have to be motivated somehow.[10]

Since I reject **Mat**, I will ultimately have to say where these arguments for **Mat** go wrong. But I first want to explain why I reject **Mat**, in the hope that this will put you in the mood to re-evaluate the arguments.

3 Higher-order Opinions

I will start by explaining my premises about unconditional sureness: why I reject **4** but accept $\mathbf{5}^{\mathbf{c}}$. Since the dialectic surrounding these principles is well trodden, I will mostly re-trace a few known ways into my position.

Why reject **4**? Following Williamson (2000), I think there are mundane counterexamples to **4**. How many typos are there in this paper? You are sure that there is at least one typoo, but fortunately not sure there are at least 1000. So there is a cut-off: a largest number n such that you are sure that there are at least n typos in this paper. You can't be sure what that cut-off is, and so in particular you can't be sure whether you're sure that there are at least n typos in this paper.[11] So **4** is false.

Not everybody is convinced by Williamson's argument. If you aren't, you may be interested to hear that having the inner and outer modalities in **4** and $\mathbf{5}^{\mathbf{c}}$ coincide isn't essential to my argument. What I *really* need are two modalities, $\Box$ and $\blacksquare$, such that the analogous principle $\mathbf{4}_{\Box\blacksquare}$ can fail but $\mathbf{5}^{\mathbf{c}}_{\Box\blacksquare}$ holds. To make this concrete, let '$\Box$' express what I am sure of, and '$\blacksquare$' express what my epistemic peer Ethan is sure of:

> $\mathbf{4}_{\Box\blacksquare}$. $\Box p \supset \Box\blacksquare p$
> If I'm sure that p, then I'm sure that Ethan is sure that p.

> $\mathbf{5}^{\mathbf{c}}_{\Box\blacksquare}$. $\Box\neg\blacksquare p \supset \neg\Box p$
> If I'm sure that Ethan isn't sure that p, then I'm not sure that p.

(2021), Rothschild & Spectre (2018), Hewson & Kirkpatrick (2022), and §9.1 below.

[8] See Pettigrew (2020) for an overview, and Teller (1973) and Greaves & Wallace (2006).

[9] See e.g. Alchourrón et al. (1985), Harper (1975, 230), Stalnaker (2009, 194).

[10] Harman (1986, 30ff.) thinks our inability to keep track of our reasons for accepting or intending is crucial in justifying minimal change, whereas Gärdenfors (1988, 49) gestures at a motivation from the thought that information does not come for free. Stalnaker (2009, 194) suggests that "to fully accept something (to treat it as knowledge) is to [...] continue accepting it unless evidence forces one to give up something."

[11] Williamson (2000, ch. 5)'s argument concerns knowledge, but generalises to sureness (Boylan & Schultheis, 2022, §IV) and further attitudes (Hawthorne & Magidor, 2009, 2010).

Even if you accept $\mathbf{5^c}$ *and* **4**, you may well accept $\mathbf{5^c_{\Box\blacksquare}}$ but reject $\mathbf{4_{\Box\blacksquare}}$. Given that disagreement among epistemic peers is common, it seems okay for me to be sure of something without being sure that Ethan is, too. After all, sometimes I am sure of things but then find out that Ethan assesses the evidence differently. However, it still seems that if I am sure that my peer *isn't* sure of something, I should not be sure of it myself. That's $\mathbf{5^c_{\Box\blacksquare}}$.

This is all I'll say against **4** and its bimodal generalisation $\mathbf{4_{\Box\blacksquare}}$. If you're unconvinced, you can read much of what follows as a new argument for these principles from **Mat**. I have slightly more to say in defence of $\mathbf{5^c}$.

First, *sureness akrasia* simply seems irrational: being sure that (p, but I shouldn't be sure that p). It's irrational to eat a mushroom when you are sure that you shouldn't. Similarly, it's irrational to be sure that this mushroom is edible when you are sure that you shouldn't.[12] (Recall that what you *should* be sure of is what I'm really interested in.)

Second, $\mathbf{5^c}$ is the weakest principle that rules out being sure of Moorean propositions: ones that can be true, but only if you are not sure of them.[13] Perhaps this explains our intuition that the akratic beliefs are irrational.

Third, $\mathbf{5^c}$ follows from familiar more general principles such as **T** ($\Box p \supset p$),[14] or the claim that you are sure of something only if you are not sure that you don't know it ($\Box p \supset \Diamond Kp$), assuming that knowledge implies being sure ($Kp \supset \Box p$).[15] If to be sure is to have credence one, $\mathbf{5^c}$ follows from the thought that when rational agents have credence one that they ought not have some opinions, they do not have those opinions,[16] or (for finite probability spaces) from Dorst (2020)'s *Simple Trust* (the constraint $P(p \mid P(p) \geq t) \geq t$, where P are the opinions you should have).[17] I do not assume any of these more general principles, including **T**, but they still suggest that $\mathbf{5^c}$ follows from popular theories of rationality.

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[12]Objection: We can't trust our intuitions about akrasia. *Confidence akrasia* also seems irrational: being *confident* that (p, but I should not be confident that p). And yet Williamson (2011) argues that confidence akrasia can be rational when p is a long conjunction of propositions for which **4** independently fails. Reply: It's not obvious that you should really be sure of such long conjunctions. In any case, it is one thing to eat a mushroom when you are merely *confident* you shouldn't, and another when you are *sure* you shouldn't.

[13]See Rieger (2015). Mackie (1980, 91), Joyce (2009, 277), Rosenkranz (2018, 327), and Smithies (2012, 285) are moved to accept $\mathbf{5^c}$ for justified belief by similar considerations.

[14]See Goodman & Holguín (2022); and Williamson (2000, 2011) for certainty.

[15]Assume $\Box$ is normal. Necessitating the contraposition of $Kp \supset \Box p$ and distributing, we get $\Box\neg\Box p \supset \Box\neg Kp$, and so by $\Box p \supset \Diamond Kp$ we infer $\Box\neg\Box p \supset \neg\Box p$. Aucher (2015), Holguín (2021, fn.34), Lenzen (1979), Rieger (2015), and Stalnaker (2006) reason in parallel for belief.

[16]See Christensen (2007, 325)'s *Accuracy* principle, and Sobel (1987, 69f.).

[17]If $P(p) = 1$ but $P(P(p) < 1) = 1$, then $P(p \mid P(p) < 1) = 1$ by the ratio formula. For P with finite domain there is guaranteed to be $\varepsilon > 0$ with $[P(p) < 1] = [P(p) \leq 1 - \varepsilon]$ (allowing us to convert $<$ into $\leq$). Hence we have $P(p \mid P(p) \leq 1 - \varepsilon) = 1 \not\leq 1 - \varepsilon$, and by the rule of subtraction $P(\neg p \mid P(\neg p) \geq \varepsilon) = 0 \not\geq \varepsilon$ contradicting Simple Trust.

4 From $\mathbf{5^c}$ to $\mathbf{C5^c}$

The most distinctive premise of my argument says that if $\mathbf{5^c}$ is true, then so is a parallel principle for conditional sureness:[18]

> $\mathbf{C5^c}$. $\Box^q \neg \Box^q p \supset \neg \Box^q p$
> If you are sure given q that you are not sure given q that p, you are not sure given q that p.

In fact, I only need $\mathbf{C5^c}$ for suppositions that don't lead to contradiction:[19]

> $\mathbf{C5^{c\perp}}$. $\neg \Box^q \bot \supset (\Box^q \neg \Box^q p \supset \neg \Box^q p)$

My proof will proceed from $\mathbf{C5^{c\perp}}$, but informally I will drop the restriction to the non-degenerate case unless it matters.

I will now argue that we should extend $\mathbf{5^c}$ to $\mathbf{C5^c}$, first from a general parallelism between conditional and unconditional attitudes, and then by showing that the motivations for $\mathbf{5^c}$ from §3 generalise to $\mathbf{C5^c}$.

4.1 Parallelism

Conditional and unconditional beliefs must have something important in common that makes them beliefs. If it is part of their causal or normative role, then unconditional and conditional beliefs must share that part of their causal or normative role, and whatever further features result from it downstream. We should expect conditional mental state types to be related to one another just like their unconditional counterparts — conditional belief, desire, and intention stand to one another just like unconditional belief, desire, and intention. We should expect what you believe conditional on p to be related to what's true *if* p the way what you unconditionally believe is related to what's true simpliciter. Most importantly for my purposes, we should expect conditional beliefs to be related to one another the way unconditional beliefs are related to one another.

In developing theories of conditional mental states, philosophers have often implicitly or explicitly assumed such parallelism. For example, the axioms for conditional probability by Popper and Rényi closely mirror the Kolmogorov axioms for unconditional probability. Joyce (1999, 234) makes it an axiom of his decision theory that conditional likelihoods and preferences obey the same rationality constraints as unconditional likelihoods and preferences.[20] Philosophers of mind often assume that conditional and

[18] Given the natural assumption that $\Box p \equiv \Box^\top p$, $\mathbf{5^c}$ is a special case of $\mathbf{C5^c}$.

[19] Unrestricted $\mathbf{C5^c}$ conflicts with Success ($\Box^p p$) and **CRM** ($q \supset r / \Box^p q \supset \Box^p r$). Thanks to [Anonymized] for pointing this out to me.

[20] See Joyce (1999, 234)'s "Conditional Rationality" axiom, and Bradley (2017, 92).

unconditional mental states are descriptively similar in important ways.[21] While it may be hard to state in a general fashion what this parallelism amounts to, I think $\mathbf{5^c}$ and $\mathbf{C5^c}$ are, in the relevant sense, parallel.[22]

Parallelism is also supported by the connection between supposing and learning. Whatever descriptive or normative generalisations apply to being sure, they would still apply if you learnt something new. In particular, if you should obey $\mathbf{5^c}$, you should still obey $\mathbf{5^c}$ if you learnt something new. $\mathbf{C5^c}$ does not *follow* from this observation since there are propositions that you can suppose true but cannot learn (Stalnaker, 1970, 71).[23] Nevertheless $\mathbf{C5^c}$ is *a good explanation* why you should obey $\mathbf{5^c}$ if you learnt something new: If the opinions you should have upon learning q are the opinions you should now have given q, and you should be sure of this upon learning q, $\mathbf{C5^c}$ predicts that you should obey $\mathbf{5^c}$ upon learning q.[24]

4.2 Mirroring

Even if you didn't like parallelism in general, you should recognize that the particular considerations favouring $\mathbf{5^c}$ generalize to $\mathbf{C5^c}$. Just like sureness akrasia, *conditional sureness akrasia* simply seems irrational: being sure given q that (p but I should not be sure given q that p).[25] Suppose you violate $\mathbf{C5^c}$: assuming this is a button mushroom, you are sure that (this mushroom is edible, but I shouldn't be sure, on this assumption, that it is edible). Though harder to parse, this is just as irrational!

And as like $\mathbf{5^c}$ is the weakest principle that rules out being sure of Moorean propositions — propositions which can be true but only if you aren't sure of them — $\mathbf{C5^c}$ is the weakest principle which rules out being

[21] "Offline" mental states are said to resemble their "online" counterparts in *character*, *functional profile*, and *neural implementation* (Currie & Ravenscroft, 2002, §1.3; Goldman, 2006, 147, 283), and to be manipulated by the same processes (Goldman, 2006, 287; Stich & Nichols, 2000; Williamson, 2020, §2.2).

[22] Blumberg & Lederman (2020, fn. 32) suggest, crediting Jeremy Goodman and Matt Mandelkern, that conditional mental states can be radically introspectively inaccessible. Say that you believe p relative to question Q iff you believe it conditional on the true answer to Q. Blumberg & Lederman observe that one can be ignorant (or mistaken) about whether one believes p relative to Q because one is ignorant (or mistaken) about the true answer to Q. Their observation is compatible with parallelism: First, the radical lack of access concerns what one believes relative to a question, not what one believes conditional on its various answers. Second, their observation suggests only that one may have false *unconstrained* beliefs about what one believes relative to a question Q, not that one may have false beliefs *relative to question* Q about what one believes relative to question Q. Their example only motivates radical failures of $\Box^Q p \supset \Box\Box^Q p$, not of $\Box^Q p \supset \Box^Q\Box^Q p$.

[23] Lasonen-Aarnio (2015, 153) points out another reason for care: you may not be sure after learning q that q is what you learnt, and so not sure that the opinions you should now have are your old ones conditional on q.

[24] Dorst (2020, 593), Elga (2013, 136), Pettigrew & Titelbaum (2014), and Ross (2006, 283) argue in parallel for extending other deference principles to conditional opinions.

[25] $\mathbf{C5^c}$ follows given the agglomeration principle that if you are sure given p that q and you are sure given p that r, you are sure given p that $q \wedge r$ ($(\Box^p q \wedge \Box^p r) \supset \Box^p(q \wedge r)$).

conditionally sure of conditionally Moorean propositions — ones which can be true, but only if you aren't conditionally sure of them (see fn. 13).

Our third batch of motivations derived $\mathbf{5^c}$ from general principles, all of which (except one) have similarly plausible analogues for conditional sureness. For example, $\mathbf{C5^c}$ follows from the claim that you're conditionally sure of something only if you aren't conditionally sure that you don't conditionally know it ($\Box^q p \supset \Diamond^q K^q p$) assuming that conditional knowledge implies conditional sureness ($K^q p \supset \Box^q p$). If sureness implied credence one, it would follow from the principle that when you're conditionally sure that your conditional opinions should be in a certain range, they really are in that range.[26] Finally, just like $\mathbf{5^c}$ follows from Dorst (2020)'s *Simple Trust*, $\mathbf{C5^c}$ follows from Dorst's *Trust*, i.e. the constraint $P_q(p \mid P_q(p) \geq t) \geq t$ (where P_q are the conditional opinions you should have).[27] (As above, this latter argument assumes finitude, and identifying being sure with credence 1.) The point is that the motivations for $\mathbf{5^c}$ and $\mathbf{C5^c}$ seem symmetric.

Of the arguments for $\mathbf{5^c}$ from §3, the only one whose analogue for $\mathbf{C5^c}$ is clearly less plausible is that from **T**. Conditional sureness may be *conditionally* factive in the sense that being sure of p given q implies $q \supset p$, but it is not factive in the sense of implying p. Conditional factivity only gives us the restriction of $\mathbf{C5^c}$ to true suppositions ($q \supset (\Box^q \neg \Box^q p \supset \neg \Box^q p)$). This opens up a way to resist my argument: We could consistently accept $\mathbf{5^c}$ as an instance of **T**, accept **Mat**, but reject $\mathbf{C5^c}$ and **4**. By accepting $\mathbf{C5^c}$ for *true* suppositions, we might hope to explain the appeal of $\mathbf{C5^c}$.

I'm personally not so attracted to this position because I don't accept $\mathbf{5^c}$ just qua instance of **T**. Like many others (see fn. 13), I think $\mathbf{5^c}$ is plausible primarily because it rules out being sure of Moorean propositions. But suppose we forget about that, how well does the restriction of $\mathbf{C5^c}$ to *true* propositions capture the appeal of the full strength principle? It *can* explain why you will still obey $\mathbf{5^c}$ after you learn something, since what's learned is presumably true. But even if you will never learn the false suppositions, you might not be sure of that, and hence make plans for how to update if you learn them. Some of the **4**-failure cases relevant to our proof are arguably like that.[28] If the opinions you plan to have upon learning a supposition are the opinions you now have conditional on it, you'll then plan to violate $\mathbf{5^c}$ if you learn the supposition. I feel that there remains something irrational in such a plan, even if it will never be actualized.

[26] Dorst (2020)'s *Reaction* principle: If $P_q(l \leq P_q(p) \leq h) = 1$ then $l \leq P_q(p) \leq h$.

[27] Assuming $\Box^q p \equiv P_q(p) = 1$, $\mathbf{C5^c}$ becomes $P_q(P_q(p) < 1) = 1 \supset P_q(p) < 1$. Suppose this fails, i.e. $P_q(p) = 1$ but $P_q(P_q(p) < 1) = 1$. Then $P_q(p \mid P_q(p) < 1) = 1$ by the ratio formula. For P_q with finite domain there is guaranteed to be $\varepsilon > 0$ with $[P_q(p) < 1] = [P_q(p) \leq 1 - \varepsilon]$ (allowing us to convert $<$ into $\leq$). Hence we have $P_q(p \mid P_q(p) \leq 1 - \varepsilon) = 1 \not\leq 1 - \varepsilon$, so by the rule of subtraction $P_q(\neg p \mid P_q(\neg p) \geq \varepsilon) = 0 \not\geq \varepsilon$ contradicting Trust.

[28] Just imagine *Flipping for Heads* from §7 so that you'll learn whether coin n was flipped.

For an analogy, consider standard arguments that you should plan to update by the ratio formula. Dutch book and accuracy dominance arguments show that you can be sure that if you plan to update in any other way you'll end up with no more money or accuracy, and you can't be sure that it won't be less (see Pettigrew, 2020). This is meant to convince us that alternative updating plans are irrational, even ones that only depart from the ratio formula if you learn something that is in fact false.[29] Such alternatives would of course never actually result in less money or accuracy, but the point is that you can't be sure they won't. By analogy, planning to violate $\mathbf{5^c}$ on certain false suppositions still strikes me as irrational if you can't be sure that you won't learn those suppositions.

5 Normality

My final and least controversial assumption is **Normality**: $\Box$ and (for any formula q) also $\Box^q$ are normal modal operators. It is a standard multi-premise closure principle for conditional and unconditional sureness.[30]

Normality has a lot going for it. Violating it for unconditional sureness would mean that for some alternative propositions you could be sure of, strictly more of them are true and no more false at any world compatible with what you're sure of (Hewson, 2021). If sureness was credence one, then **Normality** would follow from the claim that your opinions and conditional opinions should be probabilistic (given a normal logic for 'should').[31] If conditional sureness is being sure of the indicative conditional, **Normality** for $\Box^q$ follows from **Normality** for $\Box$ and the RCK rule ($p > q_1 \wedge ... \wedge p > q_n \vdash p > q$ whenever $q_1, ...q_n \vdash q$, for $n \geq 0$).

What's more, my argument doesn't require **Normality** in full strength. It would suffice to assume that if **4** can fail, it can fail for an agent obeying **Normality**. Or even that if **4** can fail, it can fail for an agent obeying the *instances* of **Normality** in my proof. Nothing in the cut-off counterexample to **4** from §3 prevents you from satisfying **Normality** (as confirmed by the usual models of such cases). Even more clearly, nothing prevents you from considering my proof, working out the relevant entailments, and hence satisfying the relevant instances of **Normality**. As Williamson (2021, 2) puts it, **4** "is not a booby prize for those who are bad at logic."

[29]Suppose your credences are c, but you aren't sure what they are, and you'll be told in an hour. The plan (for any E) to adopt $c(\cdot \mid E)$ if you learn E will in fact coincide with the plan to adopt $c(\cdot \mid C = c)$ if you learn $[C = c]$, and to adopt some random c' otherwise.

[30]See Alchourrón et al. (1985), Aucher (2015), Bradley (2017), Goodman & Salow (2025), Harper (1975), Joyce (1999), Stalnaker (2006, 2009) for **Normality**-validating theories.

[31]Suppose your credences should be probabilistic. By the normalization axiom and necessitation for 'should', you should then assign probability 1 to any tautology, ensuring necessitation for $\Box$. For **K**, we use that probability 1 is closed under finite intersection, and so also under logical consequence in the sense relevant to **K**.

6 Deriving 4 from $\mathbf{C5^{c\bot}}$, Mat, and Normality

With **Mat**, $\mathbf{C5^{c\bot}}$, and **Normality** on the table, it is time to prove that they entail **4**. In fact, as mentioned in §3, we will prove a bi-modal generalisation that allows the inner and outer modalities to differ. Consider:

$\mathbf{4}_{\Box\blacksquare}$. $\Box p \supset \Box\blacksquare p$

$\mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$. $\Box^q\neg\blacksquare^q p \supset \neg\Box^q p$

$\mathbf{C5}^{\mathbf{c}\bot}_{\Box\blacksquare}$. $\neg\Box^q\bot \supset (\Box^q\neg\blacksquare^q p \supset \neg\Box^q p)$

Using the relational semantics for modal logic, we prove that any *normal* modal logic containing all instances of **Mat** and $\mathbf{C5}^{\mathbf{c}\bot}_{\Box\blacksquare}$ (or $\mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$) contains $\mathbf{4}_{\Box\blacksquare}$. The appendix contains a syntactic proof of the same fact.

A frame $\mathfrak{F}$ is a tuple $\langle W, R_\Box, R_\blacksquare, (R^p_\Box)_{p\subseteq W}, (R^p_\blacksquare)_{p\subseteq W}\rangle$ where the R_*'s and R^p_*'s are binary relations on the non-empty set W (for $* \in \{\Box, \blacksquare\}, p \subseteq W$).[32] A model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ extends a frame $\mathfrak{F}$ with a valuation $V : At \to \mathcal{P}(W)$. We let $* \in \{\Box, \blacksquare\}$ throughout to avoid duplication. The semantic clauses for atoms and connectives are as usual, plus

- $\llbracket *p\rrbracket^w = 1$ iff $R_*(w) \subseteq \llbracket p\rrbracket$
- $\llbracket *^q p\rrbracket^w = 1$ iff $R^{\llbracket q\rrbracket}_*(w) \subseteq \llbracket p\rrbracket$

Here and later, $\llbracket p\rrbracket = \{w \in W \mid \llbracket p\rrbracket^w = 1\}$ is the set of worlds where p is true. We call p valid on a frame $\mathfrak{F}$ iff p is true at all worlds in all models $\mathfrak{M}$ that extend $\mathfrak{F}$. A class of frames M characterises a schema X when all and only the frames in M validate all instances of schema X.

As always, **Normality** is ensured by the structure of Kripke frames. **Mat** is valid on a frame iff whenever there are p-worlds in $R_*(w)$, $R^p_*(w) = R_*(w) \cap p$.[33] $\mathbf{4}_{\Box\blacksquare}$, $\mathbf{5}^{\mathbf{c}}_{\Box\blacksquare}$, and $\mathbf{5}^{\mathbf{c}\bot}_{\Box\blacksquare}$ are characterised by properties of the relations $R_\Box$ and $R_\blacksquare$ in the usual way (see e.g. Lemmon, 1977, 54):

---

[32] Similar structures were explored in an old draft of Boylan & Schultheis (2022).

[33] Proof: First let $\langle W, R_\Box, R_\blacksquare, (R^p_\Box)_{p\subseteq W}, (R^p_\blacksquare)_{p\subseteq W}\rangle$ with $R_*(w) \cap p \neq \varnothing \Rightarrow R^p_*(w) = R_*(w) \cap p$ for all $w \in W, p \subseteq W$. Extend our frame to a model, and let $w \in W$. If $R_*(w) \cap \llbracket q\rrbracket = \varnothing$, then $\llbracket *\neg q\rrbracket^w = 1$ and so trivially $\llbracket \neg*\neg q \supset (*^q p \equiv *(q \supset p))\rrbracket^w = 1$. If $R_*(w) \cap \llbracket q\rrbracket \neq \varnothing$, then $R^{\llbracket q\rrbracket}_*(w) = R_*(w) \cap \llbracket q\rrbracket$ and so $R^{\llbracket q\rrbracket}_*(w) \subseteq \llbracket p\rrbracket \Leftrightarrow (R_*(w) \cap \llbracket q\rrbracket) \subseteq \llbracket p\rrbracket$, and so $R^{\llbracket q\rrbracket}_*(w) \subseteq \llbracket p\rrbracket \Leftrightarrow R_*(w) \subseteq \llbracket q \supset p\rrbracket$, and so $\llbracket \neg*\neg q \supset (*^q p \equiv *(q \supset p))\rrbracket^w = 1$ also.

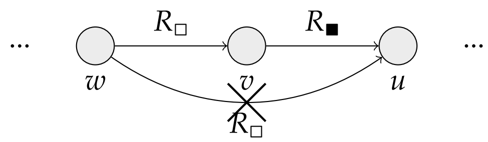
Second, consider $\langle W, R_\Box, R_\blacksquare, (R^p_\Box)_{p\subseteq W}, (R^p_\blacksquare)_{p\subseteq W}\rangle$ validating $\neg*\neg q \supset (*^q p \equiv *(q \supset p))$. Let $w \in W, p \subseteq W$ with $R_*(w) \cap p \neq \varnothing$. We consider V_1, V_2 such that $V_1(A) = V_2(A) = p$ and $V_1(B) = R_*(w) \cap p$ and $V_2(B) = R^p_*(w)$. We know $\llbracket \neg*\neg A \supset (*^A B \equiv *(A \supset B))\rrbracket^w = 1$. On both valuations we have $\llbracket \neg*\neg A\rrbracket^w = 1$, so $\llbracket *^A B \equiv *(A \supset B)\rrbracket^w = 1$ on both. This means $R^{\llbracket A\rrbracket}_*(w) \subseteq \llbracket B\rrbracket \Leftrightarrow R_*(w) \subseteq ((W \setminus \llbracket A\rrbracket) \cup \llbracket B\rrbracket)$, and so $R^{\llbracket A\rrbracket}_*(w) \subseteq V_j(B) \Leftrightarrow (R_*(w) \cap \llbracket A\rrbracket) \subseteq V_j(B)$. For V_1 this reduces to $R^p_*(w) \subseteq (R_*(w) \cap p)$, for V_2 this reduces to $(R_*(w) \cap p) \subseteq R^p_*(w)$, and so $R^p_*(w) = (R_*(w) \cap p)$.

Name	Axiom	Condition on R_*	
$\mathbf{4}_{\Box\blacksquare}$	$\Box p \supset \Box\blacksquare p$	transitive$_{\Box\blacksquare}$	$\forall xyz(R_\Box xy \wedge R_\blacksquare yz \rightarrow R_\Box xz)$
$\mathbf{5}^{\mathbf{c}}_{\Box\blacksquare}$	$\Box\neg\blacksquare p \supset \neg\Box p$	condescending$_{\Box\blacksquare}$	$\forall x\exists y(R_\Box xy \wedge \forall z(R_\blacksquare yz \rightarrow R_\Box xz))$
$\mathbf{5}^{\mathbf{c}\perp}_{\Box\blacksquare}$	$\neg\Box\perp \supset$ $(\Box\neg\blacksquare p \supset \neg\Box p)$	weakly condescending$_{\Box\blacksquare}$	$\forall x(\exists y R_\Box xy \rightarrow$ $\exists y(R_\Box xy \wedge \forall z(R_\blacksquare yz \rightarrow R_\Box xz)))$

Observe that weak condescension$_{\Box\blacksquare}$ weakens also transitivity$_{\Box\blacksquare}$. Unsurprisingly, the analogous constraints for $\Box^q$ and $\blacksquare^q$ are characterised by analogous properties of the relations $R^q_\Box$ and $R^q_\blacksquare$:[34]

Name	Axiom	Condition on R^q_*	
$\mathbf{C4}_{\Box\blacksquare}$	$\Box^q p \supset \Box^q\blacksquare^q p$	transitive$_{\Box\blacksquare}$	$\forall q\forall xyz(R^q_\Box xy \wedge R^q_\blacksquare yz \rightarrow R^q_\Box xz)$
$\mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$	$\Box^q\neg\blacksquare^q p \supset \neg\Box^q p$	condescending$_{\Box\blacksquare}$	$\forall q\forall x\exists y(R^q_\Box xy \wedge \forall z(R^q_\blacksquare yz \rightarrow R^q_\Box xz))$
$\mathbf{C5}^{\mathbf{c}\perp}_{\Box\blacksquare}$	$\neg\Box^q\perp \supset$ $(\Box^q\neg\blacksquare^q p \supset \neg\Box^q p)$	weakly condescending$_{\Box\blacksquare}$	$\forall q\forall x(\exists y R^q_\Box xy \rightarrow$ $\exists y(R^q_\Box xy \wedge \forall z(R^q_\blacksquare yz \rightarrow R^q_\Box xz)))$

These characterisations show why $\mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$ and $\mathbf{C5}^{\mathbf{c}\perp}_{\Box\blacksquare}$ entail $\mathbf{4}_{\Box\blacksquare}$ given **Mat**.[35] Suppose that **4** is invalid on a frame, and so $R_\Box$ and $R_\blacksquare$ fail to be transitive$_{\Box\blacksquare}$, i.e. there are worlds w, v, u such that $R_\Box wv$ and $R_\blacksquare vu$ but not $R_\Box wu$:



We pick our restriction as $q = \{v, u\}$ to 'zoom in' on the failure of transitivity. By the characterisation of **Mat**, since there are q-worlds in $R_\Box(w)$ and $R_\blacksquare(v)$, we know $R^q_\Box(w) = R_\Box(w) \cap q$ and $R^q_\blacksquare(v) = R_\blacksquare(v) \cap q$. We visualise $R^q_\Box$ and $R^q_\blacksquare$ by marking q black, and deleting arrows to $\neg q$-worlds:

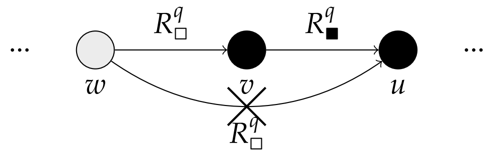
[34] I establish correspondence for $\mathbf{C5}^{\mathbf{c}\perp}_{\Box\blacksquare}$, the other proofs are similar. First consider $\mathfrak{F} = \langle W, R_\Box, R_\blacksquare, (R^p_\Box)_{p\subseteq W}, (R^p_\blacksquare)_{p\subseteq W}\rangle$ where $R^q_\Box$ and $R^q_\blacksquare$ satisfy weak condescension$_{\Box\blacksquare}$ for all $q \subseteq W$. Consider $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, and suppose $[\![\neg\Box^q\perp]\!]^w = 1$. Then $R^{[\![q]\!]}_\Box(w) \neq \varnothing$, and hence by weak condescension$_{\Box\blacksquare}$ there is $v \in R^{[\![q]\!]}_\Box(w)$ s.t. $R^{[\![q]\!]}_\blacksquare(v) \subseteq R^{[\![q]\!]}_\Box(w)$. Now if $[\![\Box^q p]\!]^w = 1$ then $R^{[\![q]\!]}_\Box(w) \subseteq [\![p]\!]$, and so $R^{[\![q]\!]}_\blacksquare(v) \subseteq [\![p]\!]$, and so $[\![\Diamond^q\blacksquare^q p]\!]^w = 1$.

Now let $\mathfrak{F} = \langle W, R_\Box, R_\blacksquare, (R^p_\Box)_{p\subseteq W}, (R^p_\blacksquare)_{p\subseteq W}\rangle$ where $R^q_\Box$ and $R^q_\blacksquare$ fail to be weakly condescending$_{\Box\blacksquare}$ for some $q \subseteq W$, i.e. there is $w \in W$ s.t. $R^q_\Box(w) \neq \varnothing$ and yet for all $v \in R^q_\Box(w)$, $R^q_\blacksquare(v) \not\subseteq R^q_\Box(w)$. Consider V s.t. $V(A) = R^q_\Box(w)$ and $V(B) = q$. Then $[\![\neg\Box^B\perp]\!]^w = 1$ since $R^q_\Box(w) \neq \varnothing$. And $[\![\Box^B A]\!]^w = 1$ for trivially $R^q_\Box(w) \subseteq R^q_\Box(w)$. But $[\![\Box^B\neg\blacksquare^B A]\!]^w = 1$ since for all $v \in R^q_\Box(w)$, $R^q_\blacksquare(v) \not\subseteq R^q_\Box(w)$, in violation of $\mathbf{C5}^{\mathbf{c}\perp}_{\Box\blacksquare}$.

[35] The characterisations also make it easy to check that $\mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$ and $\mathbf{C5}^{\mathbf{c}\perp}_{\Box\blacksquare}$ are consistent with **Mat** and **Normality**. To get models of $\mathbf{C5}^{\mathbf{c}\perp}_{\Box\blacksquare}$, start from a transitive$_{\Box\blacksquare}$ Kripke frame and define $R^p_*(w) := R_*(w) \cap p$. For $\mathbf{C5}^{\mathbf{c}}$, start from a transitive$_{\Box\blacksquare}$ Kripke frame and define

$$R^p_*(w) := \begin{cases} R_*(w) \cap p, & \text{if } R_*(w) \cap p \neq \varnothing \\ \{w\}, & \text{if } R_*(w) \cap p = \varnothing \text{ and } w \in p\text{, or } p = \varnothing \\ p, & \text{otherwise} \end{cases}$$

This makes $R^p_\Box$ and $R^p_\blacksquare$ serial and transitive$_{\Box\blacksquare}$ for all p, and so condescending$_{\Box\blacksquare}$.



Weak condescension$_{\Box\blacksquare}$ fails: $R^q_\Box(w) \neq \varnothing$ and yet for all $x \in R^q_\Box(w)$, $R^q_\blacksquare(x) \not\subseteq R^q_\Box(w)$. ($v$ is the only $x \in R^q_\Box(w)$, and $u \in R^q_\blacksquare(v)$ but $u \notin R^q_\Box(w)$.) A transitivity$_{\Box\blacksquare}$-failure for $R_\Box$ and $R_\blacksquare$ becomes a failure of weak condescension$_{\Box\blacksquare}$ once we 'zoom in' on $R^q_\Box$ and $R^q_\blacksquare$.

So **Mat** and $\mathbf{C5^{c\perp}_{\Box\blacksquare}}$ entail $\mathbf{4}_{\Box\blacksquare}$ given **Normality**. In particular, if the two modalities are identified ($\Box = \blacksquare$), **Mat** and $\mathbf{C5^{c\perp}}$ entail **4** given **Normality**. Since I reject **4** but accept $\mathbf{C5^c}$ and **Normality** for the reasons set out in §§1-5, I conclude that **Mat** has to go. In the next sections, I will explain why **4**-failures lead to **Mat**-violations, propose an alternative model of conditional sureness, and consider the implications of rejecting **Mat**.

7 Convergence with intuitions

In this section, I try to explain intuitively why **4**-failures put pressure on **Mat**, and compare the intuitions to some existing counterexamples to **Mat**.

Mat can easily be seen to be equivalent to the triad:[36]

Restricted Success. $\Diamond p \supset \Box^p p$

Preservation. $\Box p \wedge \Diamond q \supset \Box^q p$

Frontloading. $\Box^q p \supset \Box(q \supset p)$

Any counterexample to **Mat** will have to undermine one of these principles. I think **4**-failure cases put pressure on *Preservation*.

On the supposition that your unconditional opinions are misguided, it is intuitively irrational to form your conditional opinions by "minimally changing" your unconditional opinions to accommodate the supposition. After all, on this supposition you take them to be misguided.

Since **Mat** is meant to precisify "minimal change" for sureness, one would expect similar considerations to put pressure on **Mat**. While this may not have been transparent, this is indeed what my argument from the last section does. Fortunately, it turns out that it can be put in more intuitive terms if we assume **T** — that you should be sure of something only if it is true. (The official proof shows that this assumption is inessential, but at the cost of complicating the supposition considered.)

[36] *Frontloading* for knowledge is discussed by Chalmers (2012, 162), Bacon (2014, 2020); Goodman & Salow (2023), and for belief by Goodman & Salow (2025). *Preservation* is familiar from belief revision (see Gärdenfors, 1988, 157, Harper, 1975, 230).

Let's conceive of unconditional and conditional me as two people, *Un* and *Con*. *Un* is caught in a **4**-failure: *Un* should be sure that p, but should not be sure that *Un* should be sure that p. What separates *Con* from *Un* is that *Con* supposes that *Un* shouldn't be sure that p. *Con* might reason:

> **Con's reasoning**
> My supposition, that *Un* shouldn't be sure that p, cannot be evidence for p, as it follows from the negation of p by **T**.[37] So:
>
> (P1) If *Un* shouldn't be sure that p, then I shouldn't be sure that p.
> (P2) *Un* shouldn't be sure that p.
> (C) So I, *Con*, shouldn't be sure that p.

Con thus concludes that *Con* shouldn't be sure that p. But then (by $\mathbf{C5^c}$) *Con* shouldn't be sure that p. But now note that this violates *Preservation*. Since *Un* should be sure that p, and *Un* shouldn't be sure that the supposition is false, *Preservation* (and so **Mat**) says that *Con* should be sure that p, too.

Diagnosis: Where **4** fails, certain suppositions are consistent with what you should be sure of, but also make you sure that you shouldn't be sure of something that you in fact should be sure of. You then shouldn't remain sure on the suppositions in question, and so *Preservation* and **Mat** fail.

Inductive knowledge can result in similar *Preservation* failures:[38]

Flipping for Heads. 1000 fair coins were flipped one after another last night, until one landed heads or all were flipped. You know about this set-up, but have not heard anything more. In fact, the first landed heads.

If you are like me, you will be sure that not all coins were flipped. Let n be the largest x such that you are not sure that the xth coin was not flipped. By the choice of n, you are sure that the $n+1$th coin was not flipped, but you are not sure that the nth coin was not flipped. But—contra *Preservation*—on the supposition that the nth coin was flipped, you'd better not be sure that the $n+1$th coin wasn't flipped. After all, if the nth coin was flipped, it only needed to land tails and the nth coin would be flipped.

When you're sure of something that's not evidentially certain but sufficiently likely, you should no longer be sure on suppositions conditional on which it is significantly less likely. I propose that when **4** fails, the supposition that you shouldn't be sure that p is this sort of supposition.

Goodman & Salow (2025) surprisingly argue that *Frontloading* also fails in *Flipping for Heads*. (They are concerned with a version of the principle for *learning*, but I will translate their argument to the case of *supposing*.[39])

[37] This step is not obviously okay, but definitely okay if Frontloading holds.

[38] From Dorr et al. (2014), similar counterexamples to *Preservation* are discussed by Hall (1999); Goldstein & Hawthorne (2021); Goodman & Salow (2018, 2023, 2025).

[39] Frontloading holds on the theory of supposing Goodman & Salow (2025) sketch in appendix D.3, but at the cost of losing parallelism: for them, conditional sureness isn't related to conditional evidential probability the way unconditional sureness is to unconditional evidential probability.

Recall that n is the largest x such that you're not sure that coin x wasn't flipped. You have almost but not quite enough evidence to warrant being sure that coin n wasn't flipped. What should you think about coin n on the supposition that coin $n+1$ wasn't flipped? The information that coin $n+1$ wasn't flipped is additional evidence that coin n wasn't flipped. So perhaps on the supposition that coin $n+1$ wasn't flipped, you should be sure that coin n wasn't flipped. This violates *Frontloading*: While you should be sure of $[n$ not flipped$]$ given $[n+1$ not flipped$]$, you shouldn't be sure of the material conditional $[n+1$ not flipped$] \supset [n$ not flipped$]$.

These judgements are not beyond doubt, but they are intuitive and fit with natural models of the case. In the next section, I will use them as a starting point for a **Mat**-invalidating theory of conditional sureness.

8 A positive story

My main aim in this paper is not to offer a positive theory, but to argue against **Mat**. However, rejecting **Mat** with nothing to replace it would be problematic. As a proof of concept, I shall sketch a positive model of conditional opinions which invalidates **Mat** and **4** but validates $\mathbf{C5}^{\mathbf{c}\perp}$.

My model is intended to capture an interpretation of $\ulcorner$you should be sure that $p\urcorner$ roughly equivalent to $\ulcorner$you are in a position to know that $p\urcorner$, and hence has the feature that you should only ever be sure of truths. The sense of 'should' I have in mind is roughly the sense in which others maintain that you should be sure only of what you know (Goodman & Holguín, 2022), and more broadly the one operative in discussions of norms of belief. Of course even if you don't think there is such an interpretation, or are simply interested in a different interpretation, my model still establishes the consistency of the principles I have defended — a job that only gets harder by adding **T** ($\Box p \supset p$).

I assume that any proposition p has a (unique, precise) evidential probability $P(p)$ measuring how likely p is on your evidence. I take these evidential probabilities to determine what you should be sure of, but in a way where you should sometimes be sure of things that are less than evidentially certain.[40] Your conditional evidential probabilities will be assumed to analogously determine what you should be conditionally sure of:

Sureness : Evidential Probability
:: Conditional Sureness : Conditional Evidential Probability

To keep things simple, we do not allow the evidential probabilities to vary from world to world, though a more realistic model should presumably

[40] See Goodman & Salow (2021, 2023, 2025) and Goldstein & Hawthorne (2021).

allow for such variation, too. Instead, variation in what you should be sure of is, in our model, driven by variation in the underlying facts.

In line with **Normality**, I will assume that what you should be (conditionally) sure is closed under logical consequence, and can thus be represented by a set of possible worlds — those where everything you should be (conditionally) sure of is true. That set of worlds should plausibly be upward closed: if you shouldn't be sure at w that you're not in world v, and u is at least as likely on your evidence as v, then you shouldn't be sure at w that you're not in world u. (If $v \in R(w)$ and $P(\{u\}) \geq P(\{v\})$, then $u \in R(w)$. For simplicity, we take W to be finite, so we can work with probabilities rather than densities.)

Which upward closed set represents what you should be sure of? For any $w \in W$, consider the downset $\downarrow w = \{v \in W \mid P(\{w\}) \geq P(\{v\})\}$ of worlds no more likely than w.[41] Following Goodman & Salow (2021), we say that you should be sure at w that you're not at v just in case $\downarrow v$ is sufficiently less likely than $\downarrow w$. That is, for $s \in (0,1)$, we then define the strongest thing you should be sure of as

$$R(w) = \{v \in W \mid P(\downarrow v)/P(\downarrow w) \geq s\}$$

That is, you're not sure at w that you're not at v just in case the probability that you're in v or a world no more likely than it, is not much smaller than the probability that you're in w, or a world no more likely than it.

The distinctive feature of my model is that in order to define what you should be sure of conditional on p, we replace W with p and the unconditional evidential probability $P(\cdot)$ with $P_p(\cdot) = P(\cdot \mid p)$, the evidential probability conditional on p. In particular:

$$R^p(w) := \{v \in p \mid P_p(\downarrow v)/P_p(\downarrow w) \geq s\}$$

(What if $P(p) = 0$, and so $P_p(\cdot)$ is undefined? For concreteness, I shall say that $R^p(w) = \varnothing$ whenever $P(p) = 0$, but it doesn't really matter.)

This model of rational sureness is constrained. It respects **Normality** and *Success* ($\Box^p p$), and for worlds rational (conditional) sureness is upward closed in (conditional) probability. The model also validates $\mathbf{5^c}$ and $\mathbf{C5^{c\perp}}$. $\mathbf{5^c}$ is valid since **T** is. And $\mathbf{C5^{c\perp}}$ is valid since R^q is weakly condescending: if $R^q(w)$ is non-empty, then for the most likely q-world v, we automatically have that $R^q(v) \subseteq R^q(w)$ and $v \in R^q(w)$.[42] A similar argument establishes the validity of $\Box^q\Box^q p \supset \Box^q p$, a positive counterpart to $\mathbf{C5^c}$.[43] The model

[41] $\downarrow w$ is the downset of w in the poset $\langle P, \leq \rangle$, for $w \leq v := P(\{w\}) \leq P(\{v\})$.

[42] *Proof:* If $\neg\Box^q\bot$ is true at w, then $R^q(w) \neq \varnothing$. Since W is finite, we can thus find a world $v \in R^q(w)$ with $P_q(\{v\}) \geq P_q(\{u\})$ for all $u \in W$. Clearly $R^q(v) \subseteq R^q(w)$ since $P_q(\downarrow v) \geq P_q(\downarrow w)$ since $\downarrow v \supseteq \downarrow v$. So R^q is weakly condescending, and hence $\mathbf{C5^{c\perp}}$ holds.

[43] *Proof:* The principle is characterised by the property of density: if $R^q xz$ then there is $y \in R^q(x)$ with $R^q xy$ and $R^q yz$. If $R^q(w) = \varnothing$, density is vacuously satisfied. Otherwise,

also predicts that you should never be sure of p on the supposition that you shouldn't be except in degenerate cases ($\neg\Box^{\neg\Box p}\bot \supset \neg\Box^{\neg\Box p}\neg\Box p$).[44]

Though constrained, the theory invalidates **Mat** and **4**. Consider a model of *Flipping for Heads* where the evidential probabilities are just the chances. That is where w_n is the world where the nth coin lands heads, $P(\{w_n\}) = 1/2^n$ and so $P({\downarrow}w_n) = 1/2^{n-1}$. For concreteness, assume that $s = 1/64 = 1/2^6$. Then the last coin for which you shouldn't be sure that it wasn't flipped is coin 7, that is $R(w_1) = \{w_i \in W \mid i \leq 7\}$.

For the counterexample to *Preservation*, consider what you should think on the supposition that coin 7 was flipped. The supposition corresponds to the proposition $[\geq 7] := \{w_i \mid i \geq 7\}$, and

$$\begin{aligned} R^{[\geq 7]}(w_1) &= \{w_i \mid P_{[\geq 7]}({\downarrow}w_i)/P_{[\geq 7]}({\downarrow}w_1) \geq 1/64\} \\ &= \{w_i \mid 7 \leq i \leq 13\} \end{aligned}$$

Unconditionally you should be sure that coin 8 wasn't flipped but unsure if coin 7 was flipped ($R(w_1) \subseteq [\leq 7]$ but $R(w_1) \cap [\geq 7] \neq \varnothing$). And yet, on the supposition that coin 7 was flipped, you shouldn't be sure that coin 8 wasn't flipped ($R^{[\geq 7]}(w_1) = \{w_i \mid 7 \leq i \leq 13\} \not\subseteq [\leq 7]$). *Preservation* fails.

What about *Frontloading*? You should unconditionally be sure that coin 8 wasn't flipped but unsure if coin 7 was flipped ($R(w_1) = [\leq 7]$). So you should not be sure of the material conditional $[\leq 7] \supset [\leq 6]$. Still, on the supposition that coin 8 wasn't flipped, you should be sure that coin 7 wasn't flipped either ($R^{[\leq 7]}(w_1) = [\leq 6]$).[45] This is the counterexample to *Frontloading* from above. These failures of *Preservation* and *Frontloading* are of course also failures of **Mat**. (*Preservation* and *Frontloading* do still hold under certain restricted conditions, capturing the thought that these principles are good approximations in a lot situations.[46])

4 also fails. For example, you should be sure that the 8th coin isn't flipped, but you shouldn't be sure that you should be sure of this. For you

since W is finite, we can find a world $v \in R^q(w)$ with $P_q(\{u\}) \geq P_q(\{v\})$ for all $u \in W$. Clearly $R^q(w) \subseteq R^q(v)$ (since $P_q({\downarrow}w) \geq P_q({\downarrow}v)$), and so if $R^q wu$ then $R^q wv$ and $R^q vu$.

[44]If $P(\{v \mid R(v) \not\subseteq p\}) > 0$, then for all $w \in W$, $R^{\{v \mid R(v) \not\subseteq p\}}(w) \not\subseteq p$). *Proof:* Let $X := \{v \mid R(v) \not\subseteq p\}$, and let v be a maximally likely X-world, and u be a maximally likely $\neg p$-world. (Each exists by finitude, since $P(X) > 0$ and so X and $\neg p$ are non-empty.) We show that $u \in R(w)$:

$$\frac{P_X({\downarrow}u)}{P_X({\downarrow}w)} = \frac{P({\downarrow}u \cap X)}{P(X)} \cdot \frac{P(X)}{P({\downarrow}w \cap X)} = \frac{P({\downarrow}u \cap X)}{P({\downarrow}w \cap X)} \geq \frac{P({\downarrow}u \cap X)}{P({\downarrow}v \cap X)} = \frac{P({\downarrow}u)}{P({\downarrow}v)} > s$$

(We know $({\downarrow}w \cap X) \subseteq ({\downarrow}v \cap X)$ since v is maximal in X, and ${\downarrow}v \cap X = {\downarrow}v$ since $v \in X$ and x is downward closed (and by the same reasoning ${\downarrow}u \cap X = {\downarrow}u$).)

[45]$R^{[\leq 7]}(w_1) = \{v \mid P_{[\leq 7]}({\downarrow}v)/P_{[\leq 7]}({\downarrow}w_1) \geq 1/64\} = \{v \mid P_{[\leq 7]}({\downarrow}v) \geq 1/64\} = \{w_i \mid (1/2^{i-1} - 1/2^7)/(1 - 1/2^7) \geq 1/64\} = \{w_i \mid 1 \leq i \leq 6\}$.

[46]*Preservation* holds for suppositions q with $P(q \mid {\downarrow}w) < P(q \mid W \setminus R(w))$. *Frontloading* holds when $P({\downarrow}w)/P({\downarrow}v) \geq P_q({\downarrow}w)/P_q({\downarrow}v)$ for some least likely world $v \in R(w)$.

shouldn't be sure that the 7th coin isn't flipped ($R(w_1) = [\leq 7]$), and in the world where the 7th coin is flipped, you shouldn't be sure that the 8th coin isn't flipped ($R(w_7) = [\leq 13]$).[47]

Upshot: Natural models of conditional opinions validate **Normality**, **C5**$^{\mathbf{c}\perp}$, and *Restricted Success* but invalidate **4**, **Mat**, *Preservation*, and *Frontloading*. The models show not only that our package of principles is consistent, but also give us a grip on what conditional opinions might be like in the absence of **Mat**. While the particulars of how sureness is taken to be determined by evidential probability would require more motivation, I hope that the general strategy of taking conditional sureness to be determined in parallel by conditional evidential probability to be plausible.

9 Implications

I've explained why I reject **Mat**, and how I think about conditional sureness instead. In this final section, I will explore the implications of my result for indicative conditionals, the logic of knowledge, and being determined to do something, connecting it to existing literature on these topics.

9.1 Embedded Or-to-If

One way to understand conditional sureness is as being sure of the indicative conditional (which we will write as '$>$'). On this interpretation, Frontloading follows from Modus Ponens ($p > q \vdash p \supset q$) and **Normality**, and Preservation and Weak Success are the well-known principle

Embedded Or-to-If. $\Box(p \vee q) \wedge \Diamond\neg p \supset \Box(\neg p > q)$

Holguín (2021) objects to Embedded Or-to-If by showing that together with background principles WCNC ($\Diamond p \supset \neg(p > q \wedge p > \neg q)$) and Shift-Factivity ($\Box(\Box p \supset p)$) it entails (in a normal modal logic)[48] the principle

No Opposite Materials. $\Diamond p \wedge \Box(p \supset q) \supset \neg\Diamond(\Diamond p \wedge \Box(p \supset \neg q))$

But, Holguín (2021) argues, *No Opposite Materials* fails in certain natural models of **4**-failure motivated by Williamson (2000).[49]

My result structurally resembles Holguín's: both show that a principle connecting unconditional to conditional sureness (**Mat** or *Embedded Or-to-If*) plus plausible background conditions implies an introspection

[47] $R(w_7) = \{w_i \mid P(\downarrow w_i)/P(\downarrow w_7) \geq 1/2^6\} = \{w_i \mid 1/2^{(i-1)} \geq 1/2^{12}\} = \{w_i \mid i \leq 13\}$.

[48] Suppose $\Diamond p \wedge \Box(p \supset q) \wedge \Diamond(\Diamond p \wedge \Box(p \supset \neg q))$. By Embedded Or-to-If and Normality, $\Diamond p \wedge \Box(p > q) \wedge \Diamond(\Diamond p \wedge \Box(p > \neg q))$. By Shift-Factivity, $\Diamond p \wedge \Box(p > q) \wedge \Diamond(\Diamond p \wedge (p > \neg q))$. By Normality, $\Diamond((p > q) \wedge \Diamond p \wedge (p > \neg q))$. By WCNC, $\Diamond\perp$, and so $\perp$ by Normality.

[49] E.g. the model $W = \{w_1, w_2, w_3, w_4\}$ with $R(w_i) = \{w_j \mid |i - j| \leq 1\}$ and $V(p) = \{w_1, w_4\}$ and $V(q) = \{w_1\}$, where at w_2 we have $\Diamond p \wedge \Box(p \supset q) \wedge \Diamond(\Diamond p \wedge \Box(p \supset \neg q))$.

principle (**4** or *No Opposite Materials*). But there are also differences.The epistemic consequence derived in my result is stronger — **4** entails *No Opposite Materials* given **Normality**, but not the other way around.[50] Indeed, natural models of sureness, e.g. those in §8 above and the appearance-reality models from Williamson (2013), validate *No Opposite Materials* but invalidate **4**.[51]

A second difference concerns a popular weakening of *Embedded Or-to-If*, which removes the $\Box$ from the consequent:

Popular Or-to-If. $\Box(p \vee q) \wedge \Diamond\neg p \supset (\neg p > q)$

Both the strict conditional and Stalnaker (1968)'s variably strict conditional validate this principle.[52] However, reasoning similar to my result shows it to still entail $\Box p \supset (\Box\Box p \vee \Diamond\Box\Box p)$ given $\mathbf{C5^{c\perp}}$ and weak background assumptions.[53] (The background assumptions are And-to-If ($p \wedge q / p > q$), Identity ($p > p$), Modus Ponens ($p > q / p \supset q$), and **Normality**.) Since this principle is arguably not much more plausible than **4**, I think we should reject this weakening, too, assuming conditional sureness really coincides with being sure of the indicative conditional.[54]

[50] $\Box(p \supset q) \vdash_{K4} \Box\Box(p \supset q)$ by **4**, and $\Diamond p \wedge \Box(p \supset \neg q) \wedge \Box(p \supset q) \vdash_{K4} \bot$ by PC and **Normality**, and so $\Box\Box(p \supset q) \vdash_{K4} \neg\Diamond(\Diamond p \wedge \Box(p \supset \neg q))$ by **Normality**. So $\vdash_{K4} \Box(p \supset q) \supset \neg\Diamond(\Diamond p \wedge \Box(p \supset \neg q))$. For a frame validating *No Opposite Materials* but not **4**, consider $W = \{w_1, w_2, w_3\}$ with $R(w_i) = \{w_j \mid |i - j| \leq 1\}$.

[51] *No Opposite Materials* is characterised by restricted convergence: when transitivity fails ($Rxy \wedge Ryz \wedge \neg Rxz$), the middle world sees no less than the first ($R(x) \subseteq R(y)$). The models from §8 are restricted convergent because transitivity fails only for worlds ordered by probability ($Rxy \wedge Ryz \wedge \neg Rxz$ only if $P(x) > P(y) > P(z)$, implying $R(x) \subseteq R(y) \subseteq R(z)$). (*Proof:* First, suppose a restricted convergent frame has $\Diamond p \wedge \Box(p \supset q) \wedge \Diamond(\Diamond p \wedge \Box(p \supset \neg q)$ true at x. Then there are $y \in R(x)$ and $z \in R(y)$ such that $p \wedge \neg q$ is true at z and $p \supset \neg q$ is true throughout $R(y)$. But $p \wedge \neg q$ is false throughout $R(x)$, so not $z \notin R(x)$. There is also $w \in R(x)$ where $p \wedge q$ is true. By restricted convergence, $w \in R(y)$. But $p \wedge q$ is true at w, and false throughout $R(y)$. Contradiction. Now suppose that a frame isn't restricted convergent, i.e. there are x, y, z, w s.t. $Rxy \wedge Ryz \wedge \neg Rxz$ and Rxw but not Ryw. Then let $[\![p]\!] = \{w, z\}$ and $q = \{w\}$. At x, we have $\Diamond p \wedge \Box(p \supset q) \wedge \Diamond(\Diamond p \wedge \Box(p \supset \neg q))$.)

[52] The principle is explicitly accepted by Hewson & Kirkpatrick (2022), while Holguín (2021) and Rothschild & Spectre (2018) retreat to a close variant which strengthens antecedent and consequent of my principle with a box: $\Box(\Box(p \vee q) \wedge \Diamond\neg p) \supset \Box(\neg p > q)$.

[53] Let $q := (p \supset \neg\Box p)$, and suppose for a contradiction $\Box p \wedge \neg\Box\Box p \wedge \neg\Diamond\Box\Box p$. Expanding the first conjunct we have $\Box((\neg\Box p \wedge p) \vee (\Box p \wedge p))$, and so $\Box((q \wedge p) \vee (\Box p \wedge p))$ by PC and **Normality**, and hence $\Box((q > p) \vee (\Box p \wedge p))$ by And-to-If. Now we turn to the right disjunct. Combining it with the third conjunct from above ($\neg\Diamond\Box\Box p$), we get $\Box((q > p) \vee (\Box p \wedge \Diamond\neg\Box p))$ by **Normality**, and so $\Box((q > p) \vee (\Box(\neg q \vee p) \wedge \Diamond q))$. By Popular Or-to-If, this implies $\Box((q > p) \vee (q > p))$ which simplifies to (*) $\Box(q > p)$. But we have $\Box(q > q)$ by Identity and **Normality**, and so $\Box(q > (p \supset \neg\Box p))$ by rewriting. Combining with (*), we have $\Box(q > \neg\Box p)$ by **Normality**. But $(q > p) \vdash (q \supset p) \vdash p$ by MP and propositional logic, so $\neg\Box p \vdash \neg\Box(q > p)$ by **Normality**. We thus infer $\Box(q > \neg\Box(q > p))$. $\neg\Box(q > \bot)$ holds as otherwise $\Box(q > \bot) \vdash \Box\neg q \vdash \Box\neg(p \supset \neg\Box p) \vdash \Box\Box p$ in contradiction to the second conjunct. Hence we can apply $\mathbf{C5^{c\perp}}$ to infer $\neg\Box(q > p)$, contradicting (*) above.

[54] Though see Goldstein (2022), who defends the principle $\Box p \supset \Diamond\Box\Box p$ for knowledge.

9.2 Abominable conditionals

Dorst (2019) uses *Embedded Or-to-If* to argue for **4** for knowledge. Glossing over some details, Dorst's argument is this: Suppose I know that Turin is in Italy without knowing that I know this ($\Box p \wedge \neg\Box\Box p$). Then I can know that (either I know that Turin is in Italy, or Turin is in Italy) by disjunction introduction ($\Box(\Box p \vee p)$), and for all I know I don't know that Turin is in Italy ($\Diamond\neg\Box p$). Hence by *Embedded Or-to-If*, I know:

(4) #If I don't know that Turin is in Italy, Turin is in Italy. ($\neg\Box p > p$)

But that's a very weird thing to say! Since I can generally say what I know, Dorst concludes that I don't know (4), and so **4** must be true.

Because I reject *Embedded Or-to-If*, I want to resist the argument that if **4** fails then (4) is known. But we can do more. When we interpret the models from §8 in terms of knowledge, they rule out knowing p on the supposition that I don't know p in all but degenerate cases ($\neg\Box^{\neg\Box p}\bot \supset \neg\Box^{\neg\Box p} p$, see fn. 44). If knowing a conditional requires conditional knowledge, abominable conditionals such as (4) then cannot be known, and so plausibly not asserted.[55] Our models thus suggest a way for **4**-deniers to predict that abominable conditionals are unknowable.[56]

I take (4) to be a "junk conditional" like (5) and (6) — one whose antecedent is a defeater for my knowledge of its consequent.[57]

(5) Context: I'm looking at a red wall in normal conditions.
#If there is trick lighting, the wall is red.

(6) Context: n is the last coin which will be flipped for all I know.
#If coin n was flipped, it did not land tails.

Junk conditionals are unassertable because you fail to know their consequent conditional on the antecedent (Sorensen, 1988; Jackson, 1979). To predict this, we need *Preservation* to fail, as on our theory from §8.

9.3 Being determined to φ

Just as you can believe or be sure of something on a supposition, you can also intend or be determined to do something on a supposition.[58] My argument against **Mat** extends to mental states with a world-to-mind direction

[55] Or at least *asserting* conditionals sounds weird when one fails to know the consequent conditional on the antecedent (Jackson, 1979; Sorensen, 1988; Williamson, 2020).

[56] Fraser (2022) and Hewson & Kirkpatrick (2022) argue that while abominable conditionals are known when **4** fails, they are unassertable for irrelevance reasons.

[57] See Hawthorne & Isaacs (2024).

[58] See Ferrero (2009), Gibbard (2003, 48ff.), Velleman (1997). There seem to be two sorts of supposition in the practical realm. I can consider which subway to take assuming the F isn't stopping at 7th Ave, or assuming I want to get to Harlem. The former supposition seems to add to my evidence, the latter to my objectives. I will focus on the former.

of fit such. I will focus on *being determined to* because I take it to stand to *intending* roughly as *being sure* stands to *believing*. Using φ, χ, ψ as variables for non-finite clauses, and 'D' for 'you're determined to', consider

Mat$_D$. $\neg D \neg \psi \supset (D^{\psi}\varphi \equiv D(\psi \supset \varphi))$
If you aren't determined not to ψ, then you're determined to φ if you ψ just in case you're determined to either not ψ or φ.

4$_D$. $D\varphi \supset DD\varphi$
If you're determined to φ, you're determined to be determined to φ.

5$^{\mathbf{c}}_D$. $D\neg D\varphi \supset D\varphi$
If you're determined not to be determined to φ, then you do are not determined to φ.

Just as for sureness, **5**$^{\mathbf{c}}_D$ seems plausible but **4**$_D$ dubious. For one thing, **4**$_D$ but not **5**$^{\mathbf{c}}_D$ seems to require always considering whether to decide to φ when considering whether to φ. For another, consider the connection between deliberating what to do and what you will do. Writing '$\Box$' for 'you are sure that' and '$\mathcal{W}$' for 'you will', I assume that when φ is "clearly under your control", we have:[59]

Link. $D\varphi \equiv \Box\mathcal{W}\varphi$
You decide to φ iff you are sure that you will φ.

Link allows us to argue for **4**$_D$ and against **5**$^{\mathbf{c}}_D$ from parallel assumptions about sureness. Writing '$\blacksquare$' to abbreviate '$\mathcal{W}\Box$' as in 'you will be sure that', we can derive **5**$^{\mathbf{c}}_D$ from **5**$^{\mathbf{c}}_{\Box\blacksquare}$— the principle that if you are sure now that you will not be sure, you are not sure now.[60] Similarly, **4**$_D$ can fail if **4**$_{\Box\blacksquare}$ can fail for propositions about whether you will do something clearly in your control — if you can be sure that you will φ without being sure that you will be sure that you will φ.[61]

So let's suppose **5**$^{\mathbf{c}}_D$ holds but **4**$_D$ fails. My argument kicks in: if we accept **5**$^{\mathbf{c}}_D$, we should also accept its analogue **C5**$^{\mathbf{c}}_D$ for conditional decisions.[62] But **C5**$^{\mathbf{c}}_D$ and **Mat**$_D$ entail **4**$_D$ given Normality. So **Mat**$_D$ fails.

[59] Goodman & Holguín (2022, n.30) credit Kyle Blumberg for drawing this connection. The restriction is required since you're sure that you will die, but not determined to die. Alternatively, one could try linking deciding to φ to being sure that you *should* φ. See Gibbard (2003, 17): "Thinking what I ought to do amounts to deciding what to do."

[60] We assume that when φ will not be, it's not that φ will be (**Pull**. $\mathcal{W}\neg\varphi \supset \neg\mathcal{W}\varphi$). *Proof:* $D\neg D\varphi$ implies $\Box\mathcal{W}\neg\Box\mathcal{W}\varphi$ by **Link** and **Normality**, and so $\Box\neg\mathcal{W}\Box\mathcal{W}\varphi = \Box\neg\blacksquare\mathcal{W}\varphi$ by **Pull**, which in turn implies $\neg\Box\mathcal{W}\varphi = D\varphi$ by **5**$^{\mathbf{c}}_{\Box\blacksquare}$ and **Link**.

[61] If $\Box\mathcal{W}\varphi \wedge \neg\Box\blacksquare\mathcal{W}\varphi$, then $D\varphi \wedge \neg DD\varphi$ by **Link** and **Normality**, contrary to **4**$_D$.

[62] Ferrero (2009, 711f.): "conditional intentions are under exactly the same requirements as unconditional intentions."

9.4 Sureness and Cartesian Certainty

Finally, let us return to the interpretation of □ that has been my focus in this paper: being sure. On this interpretation, **Mat** says that what you're sure of on a supposition should differ as little as logic permits from what you're unconditionally sure of. You should stop being sure of things you're unconditionally sure of only if inconsistency threatens otherwise, and you should become sure of new things only if they are logical consequences of the supposition and what you should be unconditionally sure of.

But inconsistency is not the only kind of incoherence, as Harman (1986) points out. Some propositions are consistent although I shouldn't ever be sure that they are true. For example, I shouldn't ever be sure that (p but I shouldn't be sure that p).[63] Similarly, some propositions are consistent although I shouldn't ever be conditionally sure that they are true. For example, I shouldn't ever be sure on the supposition that q that (p but on the supposition that q, I shouldn't be sure that p). I reject **Mat** because when **4**, it forces you to have such consistent but incoherent conditional opinions.

Once we recognize **Mat** to fail where it forces you into incoherent conditional opinions, we should be open to the idea that it fails elsewhere, too. And indeed, **Mat** also seems to fail when you should be sure of something that is less than evidentially certain, and could hence be undermined by further experience — as in *Flipping for Heads*. On reflection, I think **4**-failure cases, and really virtually everything non-trivial that we should be sure of, is like this. Unlike Cartesian absolute certainty, what you should be sure of in the ordinary sense can be undermined by further experience.

Of course, we could nevertheless interpret '□' as 'You should be absolutely certain that', and **Mat** would then look more plausible.[64] But on such a demanding conception of certainty, the usual arguments against **4** are no longer compelling, either, since you arguably should not be absolutely certain that there are at least n typos in this paper for any $n > 0$.

My central assumption is that unconditional and conditional mental states obey "parallel" generalizations. In so far as we have a theoretical grip on conditional mental states at all, it is by means of such parallelism. But if we want parallelism, we must choose: accept **4** or deny **Mat**.

[63] Similarly, not every probability function is a possible rational credence function, contra Joyce (2009, 279)'s claim that "for any assignment of probabilities $\langle p_n \rangle$ to $\langle X_n \rangle$ it seems that a believer could, in principle, have evidence that justifies her in thinking that each X_n has p_n as its objective chance. [...] $\langle p_n \rangle$ is the rational credence function for the person." While there are possible chance functions that are certain that (p but no one should ever be certain that p), there is no possible rational credence function that is certain of this claim. Neither logical consistency nor probabilistic coherence suffices for coherence.

[64] For credence 1, **Mat** follows from the ratio formula. If absolute certainty coincides with logical truth, and conditional absolute certainty coincides with being a logical consequence of the supposition, **Mat** is in effect the deduction theorem and its converse.

A Syntactic Proof

$$\mathcal{L} ::= A \mid \neg p \mid (p \supset q) \mid \Box p \mid \blacksquare p \mid \Box^q p \mid \blacksquare^q p$$

A modal logic $\mathbf{L}$ over $\mathcal{L}$ is a set of $\mathcal{L}$-sentences containing all classical truth-functional tautologies (PC) closed under modus ponens and uniform substitution. We let $* \in \{\Box, \blacksquare\}$ to avoid duplication. A modal logic is **normal** when it is closed under necessitation for each modal operator ($p / * p$ and $p / *^q p$) and contains all instances of the **K**-axiom ($*(p \supset q) \supset (*p \supset *q)$ and $*^p(q \supset r) \supset (*^p q \supset *^p r)$).

We call p a theorem of logic $\mathbf{L}$ (write: $\vdash_{\mathbf{L}} p$) when $p \in \mathbf{L}$. We say $p_1, ..., p_n$ entail q (write: $p_1, ..., p_n \vdash_{\mathbf{L}} q$) when $\vdash_{\mathbf{L}} (p_1 \wedge ... \wedge p_n) \supset q$. Let $\mathbf{L}$ be the smallest normal modal logic containing all instances of

Frontloading$_{\blacksquare}$. $\blacksquare^q p \supset \blacksquare(q \supset p)$

Preservation$_{\Box}$. $\Box p \wedge \Diamond q \supset \Box^q p$

We use $\vdash$ for $\vdash_{\mathbf{L}}$, and prove three auxiliary theorems of $\mathbf{L}$, the first of which uses **Frontloading$_{\blacksquare}$** and the other **Preservation$_{\Box}$**:

Lemma 1. *For $q = (p \supset \neg\blacksquare p)$, $\vdash \Box^q \neg\blacksquare p \supset \Box^q \neg\blacksquare^q p$*

Proof. Contraposing **Frontloading$_{\blacksquare}$**, $\vdash \neg\blacksquare(q \supset p) \supset \neg\blacksquare^q p$. By **Normality**, $\vdash \Box^q \neg\blacksquare(q \supset p) \supset \Box^q \neg\blacksquare^q p$. For $q = (p \supset \neg\blacksquare p)$, by PC we have

$$(q \supset p) \dashv\vdash (p \supset \neg\blacksquare p) \supset p \dashv\vdash p$$

By **Normality** we thus infer $\vdash \Box^q \neg\blacksquare p \supset \Box^q \neg\blacksquare^q p$. □

Lemma 2. *For $q = (p \supset \neg\blacksquare p)$, $\Box p \wedge \neg\Box\blacksquare p \vdash \Box^q p \wedge \Box^q \neg\blacksquare p$.*

Proof. Suppose $\Box p \wedge \neg\Box\blacksquare p$.

(a) $\Box p \wedge \Diamond q$ follows by the definition of q and **Normality**, and so $\Box^q p$ by **Preservation$_{\Box}$**.

(b) Similarly, $\Box q \wedge \Diamond q$ follows by the definition of q and **Normality**, and so $\Box^q q$ by **Preservation$_{\Box}$**. This unpacks to $\Box^q(p \supset \neg\blacksquare p)$, which together with (a) yields $\Box^q \neg\blacksquare p$ by **Normality**.

□

For schemata $X_1, ...X_n$, let $\mathbf{L} + X_1 ... X_n$ be the smallest normal modal logic containing all instances of **Frontloading$_{\blacksquare}$**, **Preservation$_{\Box}$**, $X_1, ...X_n$.

C5$^{c\Diamond}_{\Box\blacksquare}$. $\Diamond q \supset (\Box^q \neg\blacksquare^q p \supset \neg\Box^q p)$

4$_{\Box\blacksquare}$. $\Box p \supset \Box\blacksquare p$

Fact 1. $\vdash_{\mathbf{L}+\mathbf{C5}^{\mathbf{c}\Diamond}_{\Box\blacksquare}} \Box p \supset \Box\blacksquare p.$

Proof. Let $q = (p \supset \neg\blacksquare p)$. We prove by contradiction:

1.	$\Box p \wedge \neg\Box\blacksquare p$	Supposition
2.	$\Box^q p \wedge \Box^q \neg\blacksquare p$	Lemma 2, 1
3.	$\Box^q p \wedge \Box^q \neg\blacksquare^q p$	Lemma 1, 2
4.	$\Diamond q$	PC, **Normality**, 1
5.	$\neg\Box^q p$	$\mathbf{C5}^{\mathbf{c}\Diamond}$, 3, 4
6.	$\bot$	PC, 2, 5

$\Box$

Corollary 1. *For* $\mathbf{L}^+$ *the smallest normal modal logic containing all instances of*

Mat. $\neg{*}\neg q \supset ({*}^q p \equiv {*}(q \supset p)) \qquad ({*} \in \{\Box, \blacksquare\})$

$\Box p \supset \Box\blacksquare p$ *is also a theorem of* $\mathbf{L}^+ + \mathbf{C5}^{\mathbf{c}\Diamond}_{\Box\blacksquare}$, $\mathbf{L}^+ + \mathbf{C5}^{\mathbf{c}\bot}_{\Box\blacksquare}$, *and* $\mathbf{L}^+ + \mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$.

$\mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$. $\Box^q\neg\blacksquare^q p \supset \neg\Box^q p$

$\mathbf{C5}^{\mathbf{c}\bot}_{\Box\blacksquare}$. $\neg\Box^q\bot \supset (\Box^q\neg\blacksquare^q p \supset \neg\Box^q p)$

Proof. **Frontloading**$_{\blacksquare}$ and **Preservation**$_{\Box}$ are special cases of **Mat**. The corollary then follows from fact 1 and the fact that $\mathbf{C5}^{\mathbf{c}}_{\Box\blacksquare}$ and $\mathbf{C5}^{\mathbf{c}\bot}_{\Box\blacksquare}$ entail $\mathbf{C5}^{\mathbf{c}\Diamond}_{\Box\blacksquare}$. (To see that $\mathbf{C5}^{\mathbf{c}\bot}_{\Box\blacksquare}$ entails $\mathbf{C5}^{\mathbf{c}\Diamond}_{\Box\blacksquare}$, note that $\Diamond q \supset (\Box^q\bot \equiv \Box(q \supset \bot))$ as an instance of **Mat**, and thus $\Diamond q \supset \neg\Box^q\bot$ by **Normality** and PC.) $\Box$

$$\mathcal{L}^\infty ::= A \mid \neg p \mid (p \supset q) \mid \Box^{q_1,\dots q_n} p \mid \blacksquare^{q_1,\dots q_n} p \qquad (n \geq 0)$$

Let $\mathbf{L}^\infty$ be the smallest normal modal logic over $\mathcal{L}^\infty$ which contains $\mathbf{Mat}^\infty$:

$\mathbf{Mat}^\infty$. $\neg {*}^{q_1 \cdots q_n} \neg r \supset ({*}^{q_1 \cdots q_n, r} p \equiv {*}^{q_1 \cdots q_n}(r \supset p)) \qquad (n \geq 0, {*} \in \{\Box, \blacksquare\})$

Fact 2. $\vdash_{\mathbf{L}^\infty + \mathbf{C5}^{\mathbf{c}\Diamond\infty}_{\Box\blacksquare}} \Box^{q_1 \cdots q_n} p \supset \Box^{q_1 \cdots q_n}\blacksquare^{q_1 \cdots q_n} p$ *for any* $n \geq 0$, *where*

$\mathbf{C5}^{\mathbf{c}\Diamond\infty}_{\Box\blacksquare}$. $\Diamond^{q_1 \cdots q_{n-1}} q_n \supset (\Box^{q_1 \cdots q_n}\neg\blacksquare^{q_1 \cdots q_n} p \supset \neg\Box^{q_1 \cdots q_n} p) \qquad (n \geq 0)$

$\mathbf{C4}^{\infty}_{\Box\blacksquare}$. $\Box^{q_1 \cdots q_n} p \supset \Box^{q_1 \cdots q_n}\blacksquare^{q_1 \cdots q_n} p \qquad (n \geq 0)$

Proof. Analogous to the proof of fact 1 and corollary 1. (Substitute ${*}^{q_1 \cdots q_{n-1}}$ for ${*}$, q_n for q throughout.) $\Box$

Corollary 2. $\mathbf{L}^\infty + \mathbf{C4}^{\infty}_{\Box\blacksquare} = \mathbf{L}^\infty + \mathbf{C5}^{\mathbf{c}\Diamond\infty}_{\Box\blacksquare}$.

Proof. From fact 2 we have that $\mathbf{L}^\infty + \mathbf{C4}^{\infty}_{\Box\blacksquare} \subseteq \mathbf{L}^\infty + \mathbf{C5}^{\mathbf{c}\Diamond\infty}_{\Box\blacksquare}$. It remains to be shown that $\mathbf{L}^\infty + \mathbf{C5}^{\mathbf{c}\Diamond\infty}_{\Box\blacksquare} \subseteq \mathbf{L}^\infty + \mathbf{C4}^{\infty}_{\Box\blacksquare}$:

1. $\Box^{q_1 \cdots q_n} p \supset \Box^{q_1 \cdots q_n}\blacksquare^{q_1 \cdots q_n} p$ $\qquad$ $\mathbf{C4}^{\infty}_{\Box\blacksquare}$

2. $\neg\Box^{q_1\cdots q_n}\bot \supset \left(\Box^{q_1\cdots q_n} p \supset \neg\Box^{q_1\cdots q_n}\neg\blacksquare^{q_1\cdots q_n} p\right)$ **Normality**$^\infty$, 1

3. $\neg\Box^{q_1\cdots q_{n-1}}\neg q_n \supset \left(\Box^{q_1\cdots q_n}\bot \equiv \Box^{q_1\cdots q_{n-1}}(q_n \supset \bot)\right)$ **Mat**$^\infty$

4. $\neg\Box^{q_1\cdots q_{n-1}}\neg q_n \supset \neg\Box^{q_1\cdots q_n}\bot$ **Normality**$^\infty$, 3

5. $\neg\Box^{q_1\cdots q_{n-1}}\neg q_n \supset \left(\Box^{q_1\cdots q_n}\neg\blacksquare^{q_1\cdots q_n} p \supset \neg\Box^{q_1\cdots q_n} p\right)$ PC, 2, 4

□

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